country, and to obtain a collection of material for illustrating prehistoric aboriginal pathology and surgery. Considerable success was met with in both directions. The anthropology of the coast was mapped out for the distance of approximately 600 miles, and some insight was obtained into that of the highlands. It was ascertained that important separate political and cultural coastal groups, such as the Chimu and the Nasca people, were no special units, anthropologically, but belonged to the same physical type as the rest of the coast population. The collections made on this trip, being selections from nearly 5000 burials, are especially valuable. Finally, the exploration made possible rich original exhibits at San Diego, covering practically the whole field of pre-Columbian Indian pathology, to which are added 60 crania showing all the forms of ancient Indian trephining. The general results of this expedition have already been published,[3] but the material collected offers a rich opportunity for further investigation.

[1] Hrdlička, A., The most ancient skeletal remains of man, *Smithsonian Rept.* for 1914, pp. 491–552, pls. i–xli.

[2] For preliminary reports on this work, see *Smithsonian Inst., Misc. Collect.*, 60 (1912); *Compte-Rendu XIV Cong. Intern. d'Anthropologie et d'Archéologie Préhist.*, Genève, 1913; and *Trans. XVIII Intern. Cong. Americanists*, London, 1914.

[3] Hrdlička, A., Anthropological work in Peru in 1913, with notes on the pathology of the ancient Peruvians. *Smithsonian Inst., Misc. Collect.*, 61, no. 18 (Publication 2264), 1914, pp. i–v, 1–69, 26 pls.

THE SECOND DERIVATIVES OF THE EXTREMAL INTEGRAL FOR A GENERAL CLASS OF PROBLEMS OF THE CALCULUS OF VARIATIONS

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1. In an earlier paper,[1] I obtained expressions for the second derivatives of the extremal-integral for the problem of minimizing the integral

$$\int F\,(x, y, x', y')dt, \qquad (1)$$

in terms of fundamental solutions of the Weierstrass form of Jacobi's differential equation for that problem.[2] In the same paper these expressions were used for deriving necessary conditions which must be satisfied by a curve which is to minimize the integral (1), if one or both endpoints are allowed to vary along a curve,[3] or if curves are admitted whose slopes possess a finite number of finite discontinuities.[4]

2. The method of differentiation employed in arriving at these re-

sults had been used before, probably first by Poisson or by Jacobi,[5] in the treatment of the integral (1) and also of more general integrals. These uses had been restricted however either to the derivation of conditions of the first order, or else to the study of special problems of variable endpoints.[6] Now it is evident that the number of special problems of this character increases rapidly as one proceeds to consider problems involving more and more unknown functions. To secure a general mode of treatment for all these special cases, it has seemed desirable to secure formulas for the second derivatives of the extremal-integral in more general problems than the one treated in my paper cited above.

3. The present paper derives such formulae for the integrals

$$\int f(x, y_1, \cdots, y_n; y_1' \cdots, y_n')\, dx, \quad \text{where } y' = dy/dx, \tag{2}$$

and

$$\int F(y_1, \cdots, y_n; y_1', \cdots, y_n')\, dt, \quad \text{where } y' = dy/dt. \tag{3}$$

In the case of the integral (2) the classical theory suffices to carry the work through and one secures expressions for the second derivatives of the extremal-integral in terms of two sets of conjugate solutions of the self-adjoint system of Jacobi differential equations:

Denoting by u_{jk} and v_{jk} systems of solutions of Jacobi's differential equations, which satisfy the initial conditions

$$u_{jk}(x_1) = \delta_{jk},\ u_{jk}(x_2) = 0;\ v_{jk}(x_1) = 0,\ v_{jk}(x_2) = \delta_{jk},$$

and using the current abbreviated notation:

$$f_0 = \frac{\partial f}{\partial x},\ f_{n+i} = \frac{\partial f}{\partial y'_i},\ \text{etc.};\ f_0^{(r)} = f^0(x_r),\ \text{etc.};\ y'_{ri} = y'_i(x_r) = \frac{d}{dx} y_i(x)\Big|_{x=x_r},$$

we have

$$\frac{\partial^2 I}{\partial x_1 \partial x_2} = -\sum_{ijk} f^{(1)}_{n+i,\,n+j}\, y'_{1i} y'_{2k} v'_{kj}(x_1) = \sum_{ijk} f^{(2)}_{n+i,\,n+j}\, y'_{2i} y'_{1k} u'_{kj}(x_2),$$

$$\frac{\partial^2 I}{\partial x_r^2} = (-1)^r \Big[f_0^{(r)} - \sum_i f^{(r)}_{0,\,n+i}\, y'_{ri} - \sum_{ij} f^{(r)}_{n+i,\,n+j}\, y'_{ri} y''_{rj} + \sum_{ijk} f^{(r)}_{n+i,\,n+j}\, y'_{ri} y'_{rk} [\delta_{r1} u'_{kj}(x_r) + \delta_{r2} v'_{kj}(x_r)]\Big],$$

$$\begin{aligned}\frac{\partial^2 I}{\partial x_r \partial y_{ri}} &= (-1)^r \Big[f_i^{(r)} - \sum_j f^{(r)}_{n+j,\,i}\, y'_{rj} - \sum_{jk} f^{(r)}_{n+j,\,n+k}\, y'_{rj} [\delta_{r1} u'_{ik}(x_r) + \delta_{r2} v'_{ik}(x_r)]\Big] \\ &= (-1)^r \Big[f^{(r)}_{0,\,n+i} + \sum_j f^{(r)}_{n+i,\,n+j}\, y''_{rj} - \sum_{jk} f^{(r)}_{n+i,\,n+j}\, y'_{rk} [\delta_{r1} u'_{kj}(x_r) + \delta_{r2} v'_{kj}(x_r)]\Big],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 I}{\partial x_r \partial y_{si}} &= (-1)^{r-1} \sum_{jk} f^{(r)}_{n+j,\,n+k}\, y'_{rj} [\delta_{r1} v'_{ik}(x_r) + \delta_{r2} u'_{ik}(x_r)] \\ &= (-1)^r \sum_{jk} f^{(s)}_{n+i,\,n+j}\, y'_{rk} [\delta_{r1} u'_{kj}(x_s) + \delta_{r2} v'_{kj}(x_s)],\end{aligned}$$

$$\frac{\partial^2 I}{\partial y_{ri}\,\partial y_{rj}} = (-1)^r [f^{(r)}_{n+i,j} + \sum_k f^{(r)}_{n+i,\,n+k} [\delta_{r1} u'_{jk}(x_r) + \delta_{r2} v'_{jk}(x_r)]],$$

$$\frac{\partial^2 I}{\partial y_{ri}\,\partial y_{sj}} = (-1)^r \sum_k f^{(r)}_{n+i,\,n+k} [\delta_{r1} v'_{jk}(x_r) + \delta_{r2} u'_{jk}(x_r)],$$

where $i, j, k = 1, \ldots, n$; $r = 1, 2$; $r + s = 3$; and $\delta_{jk} \begin{cases} = 1 \text{ when } j = k \\ = 0 \text{ when } j \neq k. \end{cases}$

4. For the integral (3), there is not in the literature a theory of the second variation and of Jacobi's equation analogous to Weierstrass's work for the integral (1). As an aid towards securing such a theory, the author established conditions under which it is possible to reduce a set of n linearly independent linear differential forms in $n + 1$ functions to a set of n such forms of the same order in n linear combinations of these $n + 1$ functions.[7] This reduction is now used to carry through a reduction of the second variation and of Jacobi's differential equations for the integral (3), analogous to Weierstrass's reduction for the integral (1). This reduction is dependent upon the non-vanishing of one of the functions y'_i. In every regular problem one is assured that there is at every point of the interval (t_1t_2) a function y_i, whose derivative does not vanish at the point, but the function may be a different one at different points. By the use of the Heine-Borel theorem, we can then, in virtue of the continuity conditions on the functions y_i, divide the interval (t_1t_2) in a finite number of intervals in each of which the reduction is possible. We consider now the system of reduced Jacobi equations which exists in the neighborhood of the point $t = t_0$ and express the second derivatives of our extremal-integral in terms of sets of fundamental solutions of these equations. These solutions are known to exist throughout the entire interval (t_1t_2), even though the equations which they satisfy may not be valid beyond a sub-interval. In this manner a set of formulae is secured, which may be immediately applied for finding analytical conditions for a minimum of the integral (3), when one of both endpoints are allowed to vary in manifolds of $1, 2 \ldots, n - 1$ dimensions.

It is intended further to develop similar formulas for the problems of minimizing integrals (2) and (3) if the unknown functions are further conditioned so as to satisfy a system of differential equations, or a system consisting of differential and of algebraic equations.

[1] A. Dresden, *Trans. Amer. Math. Soc.*, 9, 467 (1908).

[2] See Bolza, *Vorlesungen über Variationsrechnung*, p. 233 and p. 312.

[3] Ibid., p. 316–318 and p. 328–330. These conditions had been known before.

[4] Ibid., p. 372–389.

[5] See Dienger, *Grundriss der Variationsrechnung*, p. 27.

[6] See Mayer, *Leipzig, Ber. Ges. Wiss.*, 36, 99 (1884) and 48, 436 (1896); Bliss, *Math. Ann., Leipzig*, 58, 70 (1903); Erdmann, *Zs. Math., Leipzig*, 23, 364 (1878).

[7] "On the reduction of a system of linear differential forms of any order," *Annals Mathematics*, 13, 149 (1912). In a paper entitled On the second variation, Jacobi's equation and Jacobi's theorem, *Ibid.*, 15, 78 (1913), this reduction was used to unify the Weierstrass theory of the second variation and Jacobi's equation.

GROUPS POSSESSING AT LEAST ONE SET OF INDEPENDENT GENERATORS COMPOSED OF AS MANY OPERATORS AS THERE ARE PRIME FACTORS IN THE ORDER OF THE GROUP

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If $S_1, S_2, \ldots, S_\lambda$ represent a set of operators of the group G such that these λ operators generate G but that no $\lambda-1$ of them generate G, then these λ operators are called a set of independent generators of G. Every subset of a set of independent generators of any group whatever generates a subgroup whose order contains at least as many prime factors as the number of operators in this subset. In particular the number of operators in a set of independent generators never exceeds the number of prime factors in the order of the group except in the trivial case when the group is the identity, unity not being regarded as a prime factor.

If G represents the abelian group of order, p^m and of type $(1, 1, 1, \ldots)$, it is evident that each of its possible sets of independent generators involves as many operators as there are prime factors in the order of G. In what follows we propose to determine some properties possessed by all those groups which have at least one set of independent generators composed of as many operators as there are prime factors in the order of the group. The symbol G will hereafter in the present article be used to represent any one of these groups, and we shall assume in what follows that $S_1, S_2, \ldots, S_\lambda$ always represent a set of independent generators of G such that λ is equal to the number of the prime factors of the order g of G.

Each of the operators $S_1, S_2, \ldots, S_\lambda$ must be of prime order since each subset of these independent generators generates a subgroup whose order has exactly as many prime factors as the number of operators in this subset. A necessary and sufficient condition that the groups generated by two such subsets have only the identity in common